

# The effective continuum threshold in dispersive sum rules

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We study the accuracy of the bound-state parameters obtained with the method of dispersive sum rules, one of the most popular theoretical approaches in nonperturbative QCD and hadron physics. We make use of a quantum-mechanical potential model since it provides the only possibility to probe the reliability and the accuracy of this method: one obtains the bound-state parameters from sum rules and compares these results with the exact values calculated from the Schrödinger equation. We investigate various possibilities to fix the crucial ingredient of the method of sum rules — the effective continuum threshold — and propose modifications which lead to a remarkable improvement of the accuracy of the extracted ground-state parameters compared to the standard procedures adopted in the method. Although the rigorous control of systematic uncertainties in the method of sum rules remains unfeasible, the application of the proposed procedures in QCD promises a considerable increase of the actual accuracy of the extracted hadron parameters.

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## 1. INTRODUCTION

The method of dispersive sum rules for the extraction of ground-state parameters in QCD was formulated in [1, 2] and since then has been extensively applied to the analysis of hadron properties [3].

A sum-rule calculation of hadron parameters [1, 2] involves two steps: (i) one calculates the operator product expansion (OPE) for a relevant correlator and formulates the sum rule which relates this OPE to the sum over hadronic states, and (ii) one attempts to extract ground-state parameters by a numerical procedure. Each of these steps leads to uncertainties in the final result.

The first step lies fully within QCD and allows a rigorous treatment of the uncertainties: the correlator in QCD is not known precisely (because of uncertainties in quark masses, condensates,  $\alpha_s$ , radiative corrections, etc.) but the corresponding errors in the correlator may be controlled, at least in principle. We refer to such errors as the *OPE uncertainties*.

The second step is more cumbersome: even if several terms of the OPE for the correlator were known precisely, the numerical procedures of sum rules should provide the range of values which contains the true value of the hadron parameter. We call this range the *intrinsic sum-rule uncertainty*.

In spite of the extensive applications of sum-rules in particle physics, including also flavor physics [4], where a rigorous error analysis is mandatory, a proper investigation of the systematic uncertainties of the method has been started only recently [5, 6, 7].

The method of sum rules contains a set of prescriptions which are believed to allow the control of the accuracy of the extracted bound-state parameters (see e.g. Ref. [8]). The outcome of these prescriptions is claimed to be the estimate of the *intrinsic sum-rule uncertainty*.

Obviously, the only possibility to acquire an unbiased judgement of the reliability of the error estimates in sum rules is to apply the method to a problem where the parameters of the theory may be fixed and the corresponding parameters of the ground state may be calculated independently and exactly.

Presently, only quantum-mechanical potential models provide such a possibility. A simple harmonic-oscillator (HO) potential model, used as a testing ground in [5, 6, 7], possesses the essential features of QCD — confinement and asymptotic freedom [9] — and has the following advantages: (i) the bound-state parameters (masses, wave functions, form factors) are known precisely; (ii) direct analogues of the QCD correlators may be calculated exactly.

Applying the standard sum-rule machinery, we have determined the ground-state decay constant [5] and the form factor [6] from the relevant correlators, and confronted the obtained results with the known exact values, probing in this way the accuracy of the method. We have clearly demonstrated that the standard procedures adopted in the method of sum rules do not yield realistic error estimates for the extracted ground-state parameters. Moreover, we have shown that the uncontrolled systematic errors of the form factors are typically much larger than those for the decay constants.

The natural questions which then arise are: (i) Can the “standard” procedures of the method of sum rules be modified, leading to an improvement of the extracted ground-state parameters? (ii) Can one formulate a procedure which would provide the interval surely containing the actual bound-state parameter? This would mean a rigorous control of the *intrinsic sum-rule uncertainty*.

In this Letter, we will show that the answer to the first question is “yes”, whereas the answer to the second question is “no”.

The crucial ingredient of sum rules is the effective continuum threshold  $z_c$ , which governs the accuracy of the quark-hadron duality hypothesis, the basic concept of the method. We study possible modifications of the standard procedure of fixing  $z_c$ . In the HO model, relaxing the standard assumption of a Borel-parameter independent  $z_c$  is shown to lead to a significant improvement of the extraction of the bound-state parameters, particularly, of the form factor. Even though the rigorous control over the systematic uncertainties of the ground-state parameters obtained from sum rules is not feasible (and cannot be obtained in principle in problems where the truncated OPE is the only input), the application of our findings in QCD promises a considerable improvement of the actual accuracy of the method.

## 2. HARMONIC-OSCILLATOR MODEL

We consider a non-relativistic HO model defined by the Hamiltonian ( $r \equiv |\vec{r}|$ )

$$H = H_0 + V(r), \quad H_0 = \vec{p}^2/2m, \quad V(r) = m\omega^2 r^2/2, \quad (1)$$

where all features of the bound states are calculable. For instance, for the ground (g) state one finds

$$E_g = \frac{3}{2}\omega, \quad R_g \equiv |\Psi_g(\vec{r}=0)|^2 = (m\omega/\pi)^{3/2}, \quad (2)$$

$$F_g(q) = \exp(-q^2/4m\omega),$$

where the elastic form factor of the ground state is defined according to ( $q \equiv |\vec{q}|$ )

$$F_g(q) = \langle \Psi_g | J(\vec{q}) | \Psi_g \rangle = \int d^3k \psi_g^\dagger(\vec{k}) \psi_g(\vec{k} - \vec{q}), \quad (3)$$

with the current operator  $J(\vec{q})$  given by the kernel

$$\langle \vec{r}' | J(\vec{q}) | \vec{r} \rangle = \exp(i\vec{q} \cdot \vec{r}) \delta^{(3)}(\vec{r} - \vec{r}'). \quad (4)$$

## 3. POLARIZATION OPERATOR

In the method of dispersive sum rules the basic quantity needed for the extraction of the decay constant (i.e., of the ground-state wave function at the origin) is the correlator of two currents [1]. Its quantum-mechanical analogue is

$$\Pi(T) = \langle \vec{r}_f = 0 | e^{-HT} | \vec{r}_i = 0 \rangle, \quad (5)$$

where  $T$  is the Euclidean time. In the case of the HO potential the correlator  $\Pi(T)$  is exactly known:

$$\Pi(T) = \left( \frac{m\omega}{2\pi \sinh(\omega T)} \right)^{3/2}, \quad (6)$$

$$\Pi_0(T) = \left( \frac{m}{2\pi T} \right)^{3/2},$$

$$\begin{aligned} \Pi_{\text{power}}(T) &\equiv \Pi(T) - \Pi_0(T) \\ &= \left( \frac{m}{2\pi T} \right)^{3/2} \left[ -\frac{1}{4}\omega^2 T^2 + \dots \right]. \end{aligned}$$

## 4. VERTEX FUNCTION

The basic quantity for the extraction of the form factor in the method of dispersive sum rules is the correlator of three currents [2]. The analogue of this quantity in quantum mechanics is [6]

$$\Gamma(\tau_2, \tau_1, q) = \langle \vec{r}_f = 0 | e^{-H\tau_2} J(\vec{q}) e^{-H\tau_1} | \vec{r}_i = 0 \rangle, \quad (7)$$

with the operator  $J(\vec{q})$  being defined in (4). In the HO model the exact analytic expression for  $\Gamma(\tau_2, \tau_1, q)$  was obtained in Ref. [6]. At equal times  $\tau_1 = \tau_2 = \frac{1}{2}T$  it takes the following form:

$$\Gamma(T, q) = \Pi(T) \exp\left(-\frac{q^2}{4m\omega} \tanh\left(\frac{\omega T}{2}\right)\right), \quad (8)$$

$$\Gamma_0(T, q) = \Pi_0(T) \exp\left(-\frac{q^2 T}{8m}\right),$$

$$\begin{aligned} \Gamma_{\text{power}}(T, q) &= \Gamma(T, q) - \Gamma_0(T, q) \\ &= \left( \frac{m}{2\pi T} \right)^{3/2} \left[ -\frac{1}{4}\omega^2 T^2 + \frac{q^2 \omega^2}{24m} T^3 + \dots \right]. \end{aligned}$$

In this work we will take into account all the terms in the square brackets for both  $\Pi_{\text{power}}(T)$  and  $\Gamma_{\text{power}}(T, q)$ . Notice that each term is a power in  $T$  and/or  $q^2$ . Thus, retaining a fixed number of power corrections restricts the convergence of  $\Pi_{\text{power}}(T)$  and  $\Gamma_{\text{power}}(T, q)$  to the region of not too large values of  $T$  and/or  $q^2$ , as it happens in QCD when the OPE series is truncated.

## 5. GROUND-STATE PARAMETERS

Making use of the quark-hadron duality hypothesis, which assumes that the excited-state contribution is dual to the high-energy region of the free-quark diagrams, one gets the sum rules for  $R_g$

$$R_g e^{-E_g T} = \Pi_{\text{power}}(T) + \int_0^{z_{\text{eff}}^\Pi(T)} dz \rho_0(z) e^{-zT}, \quad (9)$$

and for the form factor  $F_g(q)$

$$\begin{aligned} R_g F_g(q) e^{-E_g T} &= \Gamma_{\text{power}}(T, q) \\ &+ \int_0^{z_{\text{eff}}^\Pi(T, q)} dz_1 \int_0^{z_{\text{eff}}^\Pi(T, q)} dz_2 e^{-\frac{1}{2}(z_1+z_2)T} \Delta_0(z_1, z_2, q), \end{aligned} \quad (10)$$

where  $\rho_0(z)$  and  $\Delta_0(z_1, z_2, q)$  are the known spectral densities of the two- and three-point Feynman diagrams of the non-relativistic field theory [5, 6].

The relations (9,10) constitute the definitions of the exact effective continuum thresholds  $z_{\text{eff}}^\Pi(T)$  and  $z_{\text{eff}}^\Pi(T, q)$ . Their full  $T$ - and  $q$ -dependences can be obtained by solving Eqs. (9,10) using the exact bound-state parameters  $R_g$  and  $F_g(q)$  as well as the exact power expansions

$\Pi_{\text{power}}(T)$  and  $\Gamma_{\text{power}}(T, q)$ . In the HO model this can be easily done numerically. Without loss of generality we set  $m = \omega$  and show the corresponding results in Fig. 1. It can clearly be seen that the effective continuum threshold  $z_{\text{eff}}(T, q)$  does depend upon both  $T$  and  $q$ .

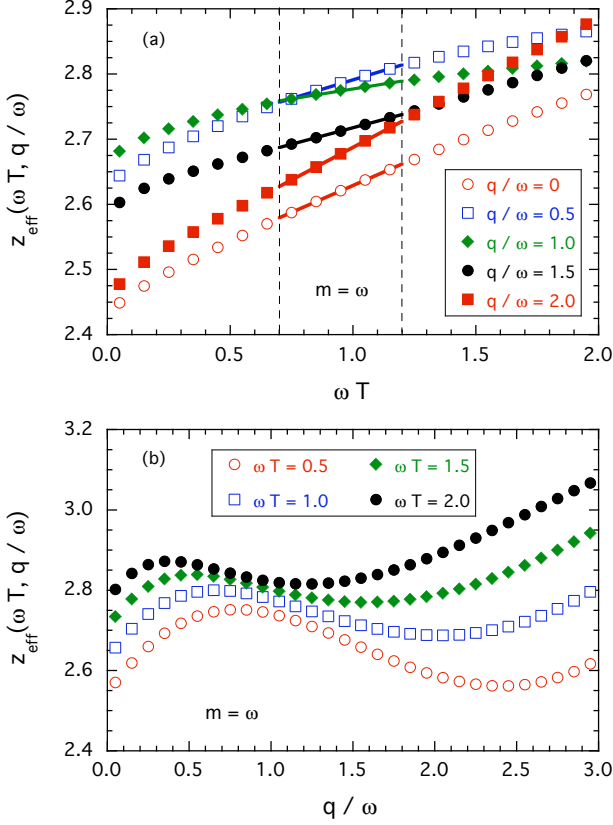


Fig. 1: Effective continuum threshold  $z_{\text{eff}}(T, q)$  for the 3-point function, obtained by solving numerically Eq. (10) using the exact bound-state parameters  $R_g$  and  $F_g(q)$  as well as the exact power expansion  $\Gamma_{\text{power}}(T, q)$ , versus the Euclidean time  $T$  at fixed values of the momentum transfer  $q$  (a) and versus  $q$  at fixed values of  $T$  (b). In (a) the vertical dashed lines identify the *fiducial range* in  $T$  (see text), while the solid lines are linear fits of  $z_{\text{eff}}(T, q)$  in the fiducial range.

Let us consider a restricted problem when the energy  $E_g$  of the ground state is known, and try to determine its elastic form factor from the sum rule (10).

First, according to [1] we should determine the Borel window (or the *fiducial range*), where the sum rule may be used for the extraction of the ground-state parameter: i) the lower boundary of the  $T$ -window is found from the requirement that the ground state gives a sizable (we require more than 50%) contribution to the correlator; and ii) the upper boundary of the  $T$ -window is obtained from the condition that the truncated OPE gives a good approximation to the exact correlator. Since in the HO model the power corrections are exactly known, the upper boundary is  $T = \infty$ . However, to be close to realistic situations when only a limited number of power corrections is available, we take from our study of Ref. [6] the

*fiducial range*  $0.7 \lesssim \omega T \lesssim 1.2$  (see Fig. 1(a)) [10].

Second, we must choose a criterion to approximate the effective continuum threshold  $z_{\text{eff}}(T, q)$ . In this work we compare three different approximations:

$$z_{\text{eff}}(T, q) \approx z_0^C(q), \quad (11)$$

$$z_{\text{eff}}(T, q) \approx z_0^L(q) + z_1^L(q) \omega T, \quad (12)$$

$$z_{\text{eff}}(T, q) \approx z_0^Q(q) + z_1^Q(q) \omega T + z_2^Q(q) \omega^2 T^2. \quad (13)$$

The standard procedure adopted in the sum-rule method is to assume a  $T$ -independent value, i.e. Eq. (11).

At each value of  $q$  we fix the parameters appearing on the r.h.s of Eqs. (11–13) in the following way: we define the dual energy,  $E_{\text{dual}}(T, q)$ , as

$$E_{\text{dual}}(T, q) = -\frac{d}{dT} \log \Gamma_{\text{dual}}(T, q, z_{\text{eff}}(T, q)), \quad (14)$$

where  $\Gamma_{\text{dual}}$  is the r.h.s. of Eq. (10) calculated using the approximations (11–13) for  $z_{\text{eff}}(T, q)$ . Then we calculate  $E_{\text{dual}}(T, q)$  at several values of  $T = T_i$  ( $i = 1, \dots, N$ ) chosen uniformly in the fiducial range and finally we minimize the squared difference with the exact value  $E_g$ :

$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [E_{\text{dual}}(T_i, q) - E_g]^2. \quad (15)$$

The results for the *dual* form factor  $F_{\text{dual}}(q)$ , obtained via Eqs. (9) and (10) using the correlator  $\Gamma_{\text{dual}}$  after optimizing the parameters of the three approximations (11–13), are shown in Fig. 2. Note that because of current conservation the form factor should obey the absolute normalization  $F_{\text{dual}}(q = 0) = 1$ . We therefore require  $z_{\text{eff}}^{\Pi}(T) = z_{\text{eff}}(T, q = 0)$ , so that the r.h.s. of Eqs. (9,10) coincide and the dual form factor is properly normalized.

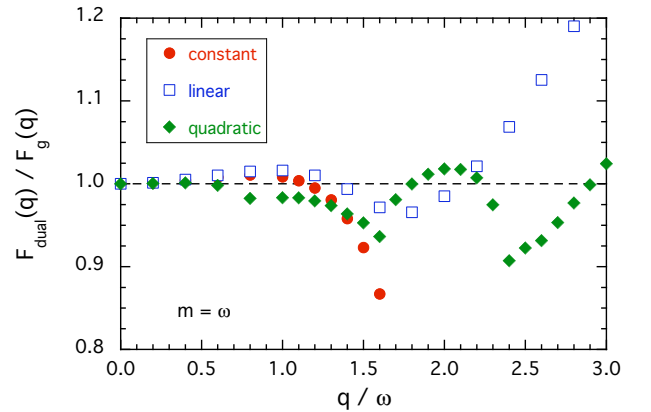


Fig. 2: Ratio of the dual form factor  $F_{\text{dual}}(q)$ , extracted from the sum rule (10) using different approximations for  $z_{\text{eff}}(T, q)$ , and the exact ground-state form factor  $F_g(q)$ , given by Eq. (2). The dots, squares and diamonds correspond, respectively, to the results obtained using the constant (11), the linear (12) and the quadratic (13) approximations for the  $T$ -dependence of  $z_{\text{eff}}(T, q)$ .

Let us consider first the case of the  $T$ -independent approximation (11). The criterion of minimizing the  $\chi^2$  (15)

leads to  $z_0^C(q)$  for which the dual energy  $E_{\text{dual}}(T, q)$  differs from  $E_g$  by less than 0.1% and to the dual form factor  $F_{\text{dual}}(q)$  which is largely  $T$ -independent in the whole fiducial range. Such a stability, usually referred to as the Borel stability, is often (erroneously) claimed to be the way to control the accuracy of the extracted form factor. From Fig. 2 it can be seen that the  $T$ -independent approximation (11) works well (better than 2%) at low values of  $q$ . However, for  $q \gtrsim 1.7\omega$  the  $T$ -independent ansatz does not work at all, since it is impossible to reproduce the ground-state energy in the fiducial range.

When the linear approximation (12) for  $z_{\text{eff}}(T, q)$  is considered, the form factor can be extracted also for  $q \gtrsim 1.7\omega$ . Note that the *exact* effective continuum threshold can be very well approximated by a linear function of  $T$  in the whole fiducial range [see Fig. 1(a)]. Nevertheless, this is not a guarantee that one can extract the *exact* form factor: deviations of the order of several percent can be produced after minimization of Eq. (15) up to  $q \approx 2\omega$  and uncertainties of the order of  $10 \div 20\%$  may plague the extracted form factor at  $q \gtrsim 2\omega$ .

One may try to go further and consider the quadratic Ansatz for  $z_{\text{eff}}(T, q)$  (13). However, as can be seen from Fig. 2, this leads to certain instabilities in the extracted value of the form factor. These instabilities just reflect the fact that the unique solution to the problem of extracting the form factor from the correlator in a limited  $T$ -window does not exist [5]. Therefore, there is no way to get a systematic improvement in the accuracy of the extracted form factor by increasing the degree of the polynomial Ansatz for  $z_{\text{eff}}(T, q)$ .

Nevertheless, it is worth emphasizing that in the HO model the comparison between the form factor extracted assuming Eq. (12) and the one obtained using Eq. (13) gives a realistic estimate of the accuracy in a wide range of values of  $q$ . Whether this feature persists in QCD is an interesting and important issue to be addressed in the future.

## 6. CONCLUSIONS

Let us summarize the main messages of our analysis:

- The knowledge of the correlator in a limited range of

relatively small Euclidean times  $T$  (that is, large Borel masses) is not sufficient for the determination of the ground-state parameters. In addition to the OPE for the relevant correlator, one needs an independent criterion for fixing the effective continuum threshold.

- Assuming a  $T$ -independent (i.e., a Borel-parameter independent) effective continuum threshold the error of the extracted ground-state parameter (both decay constant and form factor) turns out to be typically much larger than (i) the error of the description of the exact correlator by the truncated OPE and (ii) the variation of the bound-state parameter in the fiducial range (i.e., Borel window). The latter point is of particular relevance since the Borel stability is usually believed to control the accuracy of the extracted ground-state parameter. Obviously, this is not the case (see also Refs. [5, 6, 7]).

- Allowing for a  $T$ -dependent effective continuum threshold and fixing it according to Eq. (15) leads to evident improvements in the extracted ground-state parameters. This was shown for the decay constant and the form factor in the HO model. Moreover, in this model the variation of the form factor extracted using different approximations for  $z_{\text{eff}}$  gives *de facto* a realistic error estimate. Unfortunately, the use of higher polynomial approximations leads to instabilities in the fitting procedures. It is, therefore, impossible to construct a systematic procedure which would converge to the exact effective continuum threshold. As the result, rigorous error estimates cannot be obtained.

The impossibility to get a rigorous control over the systematic errors of the extracted ground-state parameters is the weak feature of the method of sum rules and an obstacle for using the results from sum rules in problems where rigorous error estimates are required.

In spite of this weakness, the application of the proposed modifications of the method in QCD seems very promising and may lead to a considerable increase of the *actual* accuracy of the calculated hadron parameters. This issue deserves a serious investigation.

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- [1] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979).
  - [2] B. L. Ioffe and A. V. Smilga, Phys. Lett. B **114**, 353 (1982). V. A. Nesterenko and A. V. Radyushkin, Phys. Lett. B **115**, 410 (1982).
  - [3] Ref. [1] with about 3500 citations is among the top cited papers in high energy physics.
  - [4] M. Artuso *et al.*, Eur. Phys. J. C **57**, 309 (2008).
  - [5] W. Lucha, D. Melikhov and S. Simula, Phys. Rev. D **76**, 036002 (2007); Phys. Lett. B **657**, 148 (2007); Phys. Atom. Nucl. **71**, 1461 (2008) [Yad. Fiz. **71**, 1489 (2008)].
  - [6] W. Lucha, D. Melikhov and S. Simula, Phys. Lett. B **671**, 445 (2009).
  - [7] D. Melikhov, Phys. Lett. B **671**, 450 (2009).
  - [8] M. Jamin and B. O. Lange, Phys. Rev. D **65**, 056005 (2002). P. Ball and R. Zwicky, Phys. Rev. D **71**, 014015 (2005).
  - [9] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov and M. A. Shifman, Sov. J. Nucl. Phys. **32**, 840 (1980) [Yad. Fiz. **32**, 1622 (1980)].
  - [10] We have checked that our findings are not sensitive to the specific choice made for the Borel window.